On the force between an electrically polarizable atom and a magnetically polarizable one

On the force between an electrically polarizable atom and a magnetically polarizable one

C Farina, F C Santos and A C Tort[1]

Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, CEP 21945-970, Rio de Janeiro, Brazil

E-mail: farina@if.ufrj.br, filadelf@if.ufrj.br, tort@if.ufrj.br and visit11@ieec.fcr.es

Abstract

In this paper we re-obtain the retarded as well as the non-retarded force between an electrically polarizable atom and a magnetically polarizable one in a very simple and suggestive way. Using a perturbative approach, we show that while the retarded force between these two atoms is proportional to $1/r^8$, the non-retarded force is proportional to $1/r^5$, in agreement with Feinberg and Sucher's result (Feinberg G and Sucher J 1970 *Phys. Rev.* A **2** 2395). This rather surprising result should be contrasted with the $1/r^7$ behaviour of the London–van der Waals force between two electrically polarizable atoms. Our approach permits us to give a physical interpretation for such a result.

PACS numbers: 12.20.−m, 32.10.Dk

Non-retarded dispersive forces between two electrically polarizable atoms were calculated in detail for the first time by London [2] who applied fourth-order perturbation theory in ordinary quantum mechanics and showed that this force is proportional to $1/r^7$, where r is the distance between the two atoms. However, when r is large compared with the transition wavelengths involved, retardation effects in the propagation of the electromagnetic interaction must be taken into account. Casimir and Polder [3] investigated the influence of retardation in the London–van der Waals forces making use of perturbative QED. They were motivated by a conjecture made by Verwey and Overbeek [4] who suggested that the interatomic force should fall faster than $1/r^7$ at large distances, otherwise experimental data for some colloidal systems and the theoretical predictions would not agree. Casimir and Polder found that the retarded force between two atoms is proportional to $1/r^8$. This kind of change also occurs when we deal with the force between an electrically polarizable atom and a perfectly conducting wall. While the non-retarded force, which is valid for short distances from the wall, is proportional to $1/r^4$ (basically the force between the induced atomic dipole and its image), the retarded

force is proportional to $1/r^5$ [3, 5]. Thus, retardation effects seem to change the exponent in the power law of the force only by one unit.

In this paper, we will investigate the non-retarded force between two non-similar atoms: an electrically polarizable atom and a magnetically polarizable one. As far as the authors know, this problem was discussed only by Feinberg and Sucher [1] three decades ago, who obtained a quite unexpected result, namely: while the retarded potential between the two atoms is proportional to $1/r^7$, the non-retarded potential is proportional to $1/r^4$. This situation must be contrasted with the $1/r^6$ power law for the non-retarded potential between two electrically polarizable atoms. In [1] the authors develop a quite general theory of the van der Waals interaction, which has the advantage of being a model-independent approach. Basically, they show that the van der Waals interaction can be expressed in terms of measurable quantities which are related to the interactions of the individual systems with real photons. More specifically, since the van der Waals interaction can be viewed as arising from the exchange of two virtual photons between the two atoms, they show that this interaction can be computed in terms of the amplitude for the emission or absorption of two real photons by each atom. Here, we employ a completely different and much simpler approach to obtain the non-retarded force mentioned above which allows us to give a physical interpretation for such a result.

The retarded interaction energy $U(r)$ between two atoms endowed with both electric and magnetic polarizations was discussed in detail by Feinberg and Sucher [6], and also by Boyer [7]. The result can be cast into the form [6, 7]:

$$U(r) = [-23 (\alpha_1\alpha_2 + \beta_1\beta_2) + 7 (\alpha_1\beta_2 + \alpha_2\beta_1)] \frac{\hbar c}{4\pi r^7} \qquad (1)$$

where α_i and β_i ($i = 1, 2$) are, respectively, the (static) electric polarizability and the (static) magnetic polarizability of the ith atom. If in equation (1) we set $\beta_1 = \beta_2 = 0$, we recover Casimir and Polder's result [3].

In order to compute the non-retarded van der Waals interaction potential for the case at hand we shall use a perturbative approach totally analogous to that found in Milonni's book [5]. However, before applying it to our problem, some comments are in order: in this approach the change in the energy of a polarizable atom due to a dynamical field has the same form as if the field were a static one, except for the fact that the atomic polarizability is now a function of the frequency of each Fourier mode of the dynamical field. This can be justified because the van der Waals interaction is a low-frequency phenomenon. For the case where the distance between the atoms is much greater than the transition wavelengths (retarded case), the dominant contribution comes from the static polarizabilities, so that dispersion is irrelevant and we have a very low frequency phenomenon. On the other hand, for the non-retarded case, where the distance between the atoms is much smaller than the transition wavelengths (but greater than a few atomic diameters to avoid electronic wavefunction overlapping), we cannot neglect dispersion, but even for this case, the dominant contributions come from low frequencies so that this formalism can still be applied.

Consider then two atoms, A and B, and assume that atom A is electrically polarizable, while atom B is magnetically polarizable. In obvious notation, the change in the energy level of atom A is given by [5]

$$\langle 0|\hat{W}|0\rangle = -\frac{1}{2} \sum_\sigma \alpha_A(\omega_\sigma) \langle 0|E_\sigma^2(x_A, t)|0\rangle. \qquad (2)$$

In the last equation $\sigma = (k, \lambda)$ characterizes the electromagnetic field mode (wave vector and polarization state, respectively) and $\alpha_A(\omega_\sigma)$ is the electric polarizability of atom A at frequency ω_σ. The point is that such (perturbative) effective quadratic interactions are known to be appropriate for off-shell (non-resonant) processes relevant to energy shifts [8]. It is now

convenient to express each mode of the total electromagnetic field operator at the position of atom A as the sum of two contributions:

$$E_\sigma(x_A, t) = E_{0\sigma}(x_A, t) + E_{B\sigma}(x_A, t) \tag{3}$$

where the first term on the r.h.s. of equation (3) stands for the unconstrained vacuum field contribution and the second term represents the contribution due to the presence of atom B. Since we are interested in the interaction potential between the two atoms, we insert equation (3) into equation (2) and retain only the crossed terms. The dispersive van der Waals interaction potential is then identified as

$$U(r) := -\frac{1}{2} \sum_\sigma \alpha_A(\omega_\sigma) \langle 0| E_{0\sigma}^{(+)}(x_A, t) \cdot E_{B\sigma}(x_A, t) + E_{B\sigma}(x_A, t) \cdot E_{0\sigma}^{(-)}(x_A, t)|0\rangle \tag{4}$$

where we have decomposed the vacuum field into positive and negative frequencies with

$$E_{0\sigma}^{(\pm)}(x, t) = \pm i \left(\frac{2\pi\hbar\omega_\sigma}{V}\right)^{1/2} a_\sigma^{(\mp)} \exp(\mp i\omega_\sigma t) \exp(\pm i k \cdot x)\hat{\varepsilon}_\sigma \tag{5}$$

where $a_\sigma^{(-)}$ and $a_\sigma^{(+)}(0)$ are respectively the usual annihilation and creation operators of a photon with momentum k and polarization λ; $\hat{\varepsilon}_\sigma$ is the polarization vector. In order to obtain an approximate expression for the field operator $E_{B\sigma}(x_A, t)$ we first recall that the classical electric field at x_A generated by an oscillating magnetic dipole m located at x_B is given by [9]

$$E(x_A, t) = -\left\{\frac{\dot{m}(t - r/c)}{cr^2} + \frac{\ddot{m}(t - r/c)}{c^2 r}\right\} \times \hat{r} \tag{6}$$

where $r = |r| := |x_A - x_B|$. Note that the electric field of an oscillating magnetic dipole does not contain the static term, i.e. there is no term proportional to $1/r^3$ in equation (6). Observe also that the equations for the electromagnetic field operators in the Heisenberg picture are formally identical to their classical counterparts. Hence, we can obtain a good approximation for the operator field $E_{B,\sigma}(x_A, t)$ if we think of m in equation (6) as the magnetic dipole of atom B induced by the vacuum magnetic field, i.e. if we write

$$m(t) = \sum_\sigma \beta_B(\omega_\sigma)[B_{0\sigma}^{(+)}(x_B, t) + B_{0\sigma}^{(-)}(x_B, t)] \tag{7}$$

where $\beta_B(\omega_\sigma)$ is the magnetic polarizability of the atom B at frequency ω_σ. Since atom A is only electrically polarizable and atom B is only magnetically polarizable, we shall suppress from now on the subscripts A and B from α and β, respectively. The analogues of equation (5) for the vacuum magnetic field operators are given by

$$B_{0\sigma}^{(\pm)}(x, t) = \pm i \left(\frac{2\pi\hbar\omega_\sigma}{V}\right)^{1/2} a_\sigma^{(\mp)} \exp(\mp i\omega_\sigma t) \exp(\pm i k \cdot r)(\hat{k} \times \hat{\varepsilon}_\sigma). \tag{8}$$

Inserting equations (8) into (7) and the result into equation (6) we get the electric field operator $E_{B\sigma}(x_A, t)$. Substituting this expression into equation (4) we obtain the following expression for $U(r)$:

$$U(r) = \frac{2\pi\hbar}{V} \text{Re}\left[\sum_\sigma k^3 \alpha(\omega_\sigma)\beta(\omega_\sigma) \exp(-ikr) \exp[ik \cdot (x_A - x_B)]\right.$$

$$\left. \times \left(\frac{i}{(kr)^2} - \frac{1}{kr}\right)(\hat{e}_\sigma \cdot (\hat{k} \times \hat{e}_\sigma) \times \hat{r})\right]. \tag{9}$$

Passing to the continuum, i.e. using

$$\sum_{k\lambda} \rightarrow \frac{V}{(2\pi)^3} \sum_{\lambda=1}^{2} \int_0^\infty dk\, k^2 \oint d\Omega_k$$

and computing the angular integral we obtain

$$U(r) = \frac{\hbar}{\pi c^6} \int_0^\infty d\omega\, \omega^6 \alpha(\omega)\beta(\omega)\mathcal{G}\left(\frac{\omega r}{c}\right) \tag{10}$$

where we have defined

$$\mathcal{G}(x) := -\frac{\sin(2x)}{x^4} + 2\frac{\cos(2x)}{x^3} + \frac{\sin(2x)}{x^2}. \tag{11}$$

The last two equations give the general expression for the interaction potential between an electrically polarizable atom and a magnetically polarizable one. However, it is convenient to analyse the retarded and the non-retarded limits separately. For large distances compared with the atomic transition wavelengths ($r \gg c/\omega_{mn}$), as in the case of two electrically polarizable atoms [5], the vacuum field modes with large wavelengths give the dominant contribution and hence, as a first approximation, we may replace the polarizabilities $\alpha(\omega)$ and $\beta(\omega)$ by their static values. Consequently, the retarded interaction potential between these atoms is given by

$$U_R(r) = \frac{\hbar c\alpha(0)\beta(0)}{\pi r^7} \int_0^\infty dx\, x^6 \mathcal{G}(x). \tag{12}$$

Observe that the $1/r^7$ behaviour of the retarded interaction energy can already be seen from the last equation, since the remaining integral gives only an extra numerical factor. Substituting equation (11) into equation (12) and evaluating the needed integrals, we obtain

$$U_R(r) = \frac{7\hbar c\alpha(0)\beta(0)}{4\pi r^7} \tag{13}$$

which agrees with equation (1) if in this equation we set $\alpha_2 = \beta_1 = 0$. Therefore, the force in the retarded case behaves like $1/r^8$, in perfect analogy with the Casimir and Polder result for two electrically polarizable atoms.

For short distances ($r \ll c/\omega_{mn}$), a situation where the retardation effects can be neglected, one would naively expect that $U_{NR}(r) \propto 1/r^6$ (and hence $F_{NR} \propto 1/r^7$), but as we will show, this is not so. In this limit, the dominant contribution to the integral in equation (10) comes from the first term of the rhs of equation (11), so that

$$U_{NR}(r) = \frac{\hbar}{\pi c^6} \int_0^\infty d\omega\, \omega^6 \alpha(\omega)\beta(\omega)\left\{-\frac{\sin(2\omega r/c)}{(\omega r/c)^4}\right\}. \tag{14}$$

In order to evaluate this integral, we need the expressions for $\alpha(\omega)$ and $\beta(\omega)$. From perturbative quantum mechanics, it can be shown that the atomic (electric) polarizability of an atom at level n is given by [10]

$$\alpha_n(\omega) = \frac{2}{3\hbar} \sum_m \frac{\omega_{mn}|d_{mn}|^2}{\omega_{mn}^2 - \omega^2} \tag{15}$$

where d_{mn} is the transition dipole moment of atom A and ω_{mn} its transition frequencies. An analogous expression also holds for $\beta(\omega)$, if we replace d_{mn} by μ_{mn} and ω_{mn} by ω'_{mn} in equation (15), μ_{mn} being the transition dipole moment of atom B and ω'_{mn} its transition frequencies. Some comments are in order here: we are not considering diamagnetic interaction between the atoms, since the dispersion relation would be different from the one we are assuming for β; moreover, in equation (15) we have neglected the linewidths, but they do exist so that for ω equal to the resonance frequencies ($\omega = \omega_{mn}$) the real part of $\alpha_n(\omega)$ vanishes

(the same remarks hold for $\beta_n(\omega)$), and the integral in equation (14) is indeed well behaved. With this in mind, we can write

$$
\begin{aligned}
U(r) &\approx -\frac{\hbar}{\pi c^2 r^4}\,\mathrm{Im}\int_0^\infty \mathrm{d}\omega\,\omega^2\alpha(\omega)\beta(\omega)\exp(\mathrm{i}2\omega r/c)\\
&= -\frac{\hbar}{\pi c^2 r^4}\,\mathrm{Im}\int_0^\infty \mathrm{i}\,\mathrm{d}\sigma\,(\mathrm{i}\sigma)^2\alpha(\mathrm{i}\sigma)\beta(\mathrm{i}\sigma)\exp(-2\omega r/c)\\
&= \frac{\hbar}{\pi c^2 r^4}\left(\frac{2}{3\hbar}\right)^2\sum_{m,p}\omega_{mn}\omega_{pn}|\boldsymbol{d}_{mn}|^2|\mu_{pn}|^2\int_0^\infty\frac{\sigma^2\mathrm{e}^{(-2\sigma r/c)}}{\left(\sigma^2+\omega_{mn}^2\right)\left(\sigma^2+\omega_{pn}'^2\right)}\,\mathrm{d}\sigma
\end{aligned}\tag{16}
$$

where we used the Cauchy residue theorem and also equation (15) as well as the analogous equation for $\beta_n(\omega)$. Since we are investigating the short distance behaviour of $U(r)$, it is legitimate to make the approximation $\exp(-2\omega r/c)\approx 1$ in the previous integral (though σ is integrated from 0 to ∞, the integrand vanishes for large values of σ due to the powers of σ present in the denominator). Moreover, assuming for simplicity that there is a dominant transition in each atom, the above equation takes the form

$$
\begin{aligned}
V(r) &= \frac{\hbar}{\pi c^2}\left(\frac{2}{3\hbar}\right)^2\frac{|\boldsymbol{d}|^2\,|\mu|^2}{r^4}\int_0^\infty\frac{\sigma^2\,\mathrm{d}\sigma}{\left(\sigma^2+\omega_0^2\right)\left(\sigma^2+\omega_0'^2\right)}\\
&= \frac{\hbar}{2c^2 r^4}\frac{\omega_0^2\omega_0'^2\alpha\beta}{(\omega_0+\omega_0')}
\end{aligned}
$$

where ω_0 and ω_0' are respectively the dominant transition frequencies of the electrically polarizable atom and the magnetically polarizable one and with $\alpha := \omega_0^{-1}(2/3\hbar)|\boldsymbol{d}|^2$ and $\beta := \omega_0'^{-1}(2/3\hbar)|\mu|^2$ being the (static) electric and the magnetic atomic polarizabilities. This result shows that when we go from the retarded to the non-retarded regime there is a striking change from r^{-7} to r^{-4} in the dispersive interaction potential, in agreement with Feinberg and Sucher's result [1]. It is worth mentioning that, if we are concerned only with this $1/r^4$ behaviour, a semiclassical calculation based on the fluctuating dipole model can also provide this result [11] (applications of the fluctuating dipole model to the usual case can be found in [5, 12]). This result is to be compared with the case where the two atoms are only electrically polarizable, in which the change is from r^{-7} to r^{-6} in the potential. The reason for such a change can be traced back to the absence of the static term in the expression for the electric field operator created by a magnetic dipole induced by the electromagnetic vacuum field fluctuations. Had we started by computing the change in the energy level of atom B, we would have obtained the same result and the reason is analogous: there is no static term in the expression for the magnetic field operator created by an electric dipole induced by the electromagnetic vacuum field fluctuations as well.

It is also the present authors' opinion that such a change in the power law of interaction (three powers of r) deserves an experimental investigation. Recall that Tabor and Winterton [13] were able to observe experimentally the change in the power law for the usual case, where only electric polarizabilities were involved (just one power of r), but no experimentalist has addressed his attention to the unusual case treated here. A possible system where our result could be checked is the hydrogen–helium system in very low energy scattering processes with both atoms in their ground states (recall that modern techniques to manipulate cold atoms are now available). The reason for that is the following: while for the helium atom the electric polarizability is much more relevant than the magnetic polarizability, the opposite happens for the hydrogen atom, i.e. the magnetic polarizability is much more relevant than the electric polarizability, since its ground state corresponds to a hyperfine singlet. Of course, the experiment we are suggesting is completely different from Tabor and Winterton's experiment,

since the latter involved static measures and ours involves scattering cross sections (different power laws yield different scattering cross sections). However, it is quite probable that the change from the $1/r^7$ to the $1/r^4$ behaviour in the power law of the interaction will not be so abrupt as in the usual case (the distance range checked by Tabor and Winterton was from 50 Å to 300 Å, and that was enough for them to observe the change from the non-retarded to the retarded regimes). The result obtained here may also be of some relevance in the analysis of the force between two macroscopic bodies.

Acknowledgments

References

[1] Feinberg G and Sucher J 1970 *Phys. Rev.* A **2** 2395
[2] London F 1930 *Z. Phys.* **63** 245
[3] Casimir H B G and Polder D 1948 *Phys. Rev.* **73** 360
    Casimir H B G 1949 *J. Chim. Phys.* **46** 407
[4] Verwey E J W and Overbeek J T G 1948 *Theory of the Stability of Lyophobic Colloids* (Amsterdam: Elsevier)
[5] Milonni P W 1994 *The Quantum Vacuum: An Introduction to Quantum Electrodynamics* (New York: Academic)
[6] Feinberg G and Sucher J 1968 *J. Chem. Phys.* **48** 3333
[7] Boyer T H 1969 *Phys. Rev.* **180** 19
    Boyer T H 1972 *Phys. Rev.* **5** 1799
[8] Compagno G and Power E A 1988 *Phys. Rev.* A **38** 4340
[9] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)
[10] Davydov A S 1965 *Quantum Mechanics* (Oxford: Pergamon)
[11] Farina C, Santos F C and Tort A C 2002 A simple model for the non-retarded dispersive force between an electrically polarizable atom and a magnetically polarizable one *Am. J. Phys.* at press
[12] Langbein D 1974 *Theory of van der Waals Attraction (Springer Tracts in Modern Physics vol 72)* (Berlin: Springer)
[13] Tabor D and Winterton R H S 1968 *Nature* **219** 1120
    Tabor D and Winterton R H S 1969 *Proc. R. Soc.* A **312** 435